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On the composition of the distributions x_+^λ and x_+^μ

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Abstract

Let F be a distribution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The distributions $(x_+^\mu)_+^\lambda$ and $(|x|^\mu)_+^\lambda$ are evaluated for $\lambda < 0$, $\mu > 0$ and $\lambda, \lambda\mu \neq -1, -2, \ldots$.

Keywords: Distribution; Delta-function; Composition of distributions; Neutrix; Neutrix limit

1. Introduction

In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$ and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

We define the locally summable functions x_+^λ and x_-^λ for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0, \end{cases} \qquad x_-^\lambda = \begin{cases} |x|^\lambda, & x < 0, \\ 0, & x > 0. \end{cases}$$

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The distributions x_+^λ and x_-^λ are then defined inductively for $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$ by

$$(x_+^\lambda)' = \lambda x_+^{\lambda-1}, \qquad (x_-^\lambda)' = -\lambda x_-^{\lambda-1}.$$

It follows that if r is a positive integer and $-r-1 < \lambda < -r$, then

$$\langle x_+^\lambda, \varphi(x)\rangle = \int_0^\infty x^\lambda \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k\right] dx,$$

$$\langle x_-^\lambda, \varphi(x)\rangle = \int_{-\infty}^0 |x|^\lambda \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^i\right] dx$$

for arbitrary φ in $\mathcal{D}$. The distribution $|x|^\lambda$ is then defined by

$$|x|^\lambda = x_+^\lambda + x_-^\lambda.$$

In particular, if φ has its support contained in the interval $[-1, 1]$, then

$$\langle x_+^\lambda, \varphi(x)\rangle = \int_0^1 x^\lambda \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k\right] dx + \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!(\lambda+k+1)} \tag{1}$$

if $-r-1 < \lambda < -r$ and

$$\langle |x|^\lambda, \varphi(x)\rangle = \int_{-1}^1 x^\lambda \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(2k)}(0)}{(2k)!} x^{2k}\right] dx + \sum_{k=0}^{r-1} \frac{2\varphi^{(2k)}(0)}{(2k)!(\lambda+2k+1)} \tag{2}$$

if $-2r-2 < \lambda < -2r$ and $\lambda \neq -2r-1$.

We now let N be the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ \ln^r n: \quad \lambda > 0, \ r = 1, 2, \ldots,$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geqslant 1$,
(ii) $\rho(x) \geqslant 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^1 \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [2].

Definition 1. Let F be a distribution and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a,b) if

$$\underset{n\to\infty}{N\text{-lim}} \int_{-\infty}^{\infty} F_n\big(f(x)\big)\varphi(x)\,dx = \big\langle h(x), \varphi(x)\big\rangle$$

for all test functions φ with compact support contained in (a,b).

The following theorems were proved in [2] and [3], respectively.

Theorem 1. *The distributions $(x_-^\mu)_-^\lambda$ and $(x_+^\mu)_-^\lambda$ exist and*

$$\big(x_-^\mu\big)_-^\lambda = \big(x_+^\mu\big)_-^\lambda = 0$$

for $\mu > 0$ and $\lambda\mu \neq -1, -2, \ldots$ and

$$\big(x_-^\mu\big)_-^\lambda = (-1)^{\lambda\mu}\big(x_+^\mu\big)_-^\lambda = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu-1)!}\delta^{(-\lambda\mu-1)}(x)$$

for $\mu > 0$, $\lambda \neq -1, -2, \ldots$ and $\lambda\mu = -1, -2, \ldots$.

Theorem 2. *The distribution $(x_+^r)_-^{-s}$ exists and*

$$\big(x_+^r\big)_-^{-s} = \frac{(-1)^{rs+s}c_1(\rho)}{r(rs-1)!}\delta^{(rs-1)}(x)$$

for $r, s = 1, 2, \ldots$.

In the previous theorem, the distribution x_-^{-s} is defined by

$$x_-^{-s} = -\frac{(\ln x_-)^{(s)}}{(s-1)!}$$

for $s = 1, 2, \ldots$ and not as in Gel'fand and Shilov [4]. We also define the distribution x_+^{-r} by the equation

$$x_+^{-r} = \frac{(-1)^{r-1}(\ln x_+)^{(r)}}{(r-1)!}$$

for $r = 1, 2, \ldots$.

Further, we define the distributions $x_+^{-1}\ln x_+$ and $x_-^{-1}\ln x_-$ by

$$x_+^{-1}\ln x_+ = \frac{1}{2}\big(\ln^2 x_+\big)', \qquad x_-^{-1}\ln x_- = -\frac{1}{2}\big(\ln^2 x_-\big)'$$

and we define the distributions $x_+^{-r-1}\ln x_+$ and $x_-^{-r-1}\ln x_-$ by

$$x_+^{-r-1}\ln x_+ = \frac{x_+^{-r-1} - (x_+^{-r}\ln x_+)'}{r}, \qquad x_-^{-r-1}\ln x_- = \frac{x_-^{-r-1} + (x_-^{-r}\ln x_-)'}{r}$$

for $r = 1, 2, \ldots$.

The next three theorems were proved in [5], [6] and [7], respectively.

Theorem 3. *If $F_s(x)$ denotes the distribution $x^{-s}\ln|x|$, then the distribution $F_s(x^r)$ exists and*

$$F_s\left(x^r\right) = rF_{rs}(x)$$

for $r, s = 1, 2, \ldots$.

Theorem 4. *Let $F_s(x)$ denote the distribution $x_-^{-s}\ln x_-$. Then the distribution $F_{ms}(x_+^{r-p/m})$ exists and*

$$\begin{aligned}
&F_{ms}\left(x_+^{r-p/m}\right) \\
&\quad= \frac{(-1)^{ms+msr+sp}\phi(ms-1)c_1(\rho)}{(r-p/m)(msr-sp-1)!}\delta^{(msr-sp-1)}(x) \\
&\qquad+ \frac{(-1)^{ms+msr+sp+1}[\phi_1(ms)+\phi_2(ms)-\phi^2(ms)+c_2(\rho)]}{(r-p/m)(msr-sp-1)!}\delta^{(msr-sp-1)}(x)
\end{aligned}$$

for $r, s = 1, 2, \ldots$ and $m = 2, 3, \ldots$, where

$$\phi(r) = \begin{cases} \sum_{i=1}^{r} 1/i, & r \geqslant 1, \\ 0, & r = 0, \end{cases} \qquad \phi_1(r) = \begin{cases} \sum_{i=1}^{r-1} \frac{\phi(i)}{i+1}, & r = 2, 3, \ldots, \\ 0, & r = 0, 1. \end{cases}$$

$$\phi_2(r) = \sum_{i=1}^{r+1} \frac{\phi(i)}{i}, \quad r = 0, 1, 2, \ldots,$$

$$c_1(\rho) = \int_0^1 \ln t\,\rho(t)\,dt, \qquad c_2(\rho) = \int_0^1 \ln^2 t\,\rho(t)\,dt,$$

$1 \leqslant p < m$ and p and m are coprime.

Theorem 5. *The distribution $(x_+^r)^{-1}$ exists and*

$$\left(x_+^r\right)^{-1} = x_+^{-r} + (-1)^r\frac{2c_1(\rho) - r\phi(r-1)}{r!}\delta^{(r-1)}(x),$$

for $r = 1, 2, \ldots$.

We now prove

Theorem 6. *The distribution $(x_+^\mu)_+^\lambda$ exists and*

$$\left(x_+^\mu\right)_+^\lambda = x_+^{\lambda\mu} \tag{3}$$

for $\lambda < 0$, $\mu > 0$ and $\lambda, \lambda\mu \neq -1, -2, \ldots$.

Proof. We will suppose that $-s-1 < \lambda < -s$ for some non-negative integer s. Then

$$x_+^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+s+1)}\left(x_+^{\lambda+s}\right)^{(s)}.$$

We put

$$\left(x_+^\lambda\right)_n = x_+^\lambda * \delta_n(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+s+1)} x_+^{\lambda+s} * \delta_n^{(s)}(x)$$

and so

$$\frac{\Gamma(\lambda+s+1)}{\Gamma(\lambda+1)}\left(x_+^\lambda\right)_n = \begin{cases} \int_{-1/n}^{1/n}(x-t)^{\lambda+s}\delta_n^{(s)}(t)\,dt, & 1/n < x, \\ \int_{-1/n}^{x}(x-t)^{\lambda+s}\delta_n^{(s)}(t)\,dt, & -1/n \leqslant x \leqslant 1/n, \\ 0, & x < -1/n. \end{cases}$$

Then

$$\frac{\Gamma(\lambda+s+1)}{\Gamma(\lambda+1)}\left[\left(x_+^\mu\right)_+^\lambda\right]_n = \begin{cases} \int_{-1/n}^{1/n}(x^\mu-t)^{\lambda+s}\delta_n^{(s)}(t)\,dt, & 1/n < x^\mu, \\ \int_{-1/n}^{x^\mu}(x^\mu-t)^{\lambda+s}\delta_n^{(s)}(t)\,dt, & 0 \leqslant x^\mu \leqslant 1/n, \\ \int_{-1/n}^{0}(-t)^{\lambda+s}\delta_n^{(s)}(t)\,dt, & x < 0. \end{cases} \tag{4}$$

It follows that

$$\begin{aligned}
&\frac{\Gamma(\lambda+s+1)}{\Gamma(\lambda+1)} \int_{-1}^{1} x^k\left[\left(x_+^\mu\right)_+^\lambda\right]_n dx \\
&= \int_0^{n^{-1/\mu}} x^k \int_{-1/n}^{x^\mu} \left(x^\mu-t\right)^{\lambda+s}\delta_n^{(s)}(t)\,dt\,dx + \int_{n^{-1/\mu}}^{1} x^k \int_{-1/n}^{1/n} \left(x^\mu-t\right)^{\lambda+s}\delta_n^{(s)}(t)\,dt\,dx \\
&\quad + \int_{-1}^{0} x^k \int_{-1/n}^{0} (-t)^{\lambda+s}\delta_n^{(s)}(t)\,dt\,dx \\
&= \frac{n^{-(\lambda\mu+k+1)/\mu}}{\mu}\int_0^1 v^{(k+1)/\mu-1}\int_{-1}^{v}(v-u)^{\lambda+s}\rho^{(s)}(u)\,du\,dv \\
&\quad + \frac{n^{-(\lambda\mu+k+1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(u)\int_1^n v^{(k+1)/\mu-1}(v-u)^{\lambda+s}\,dv\,du \\
&\quad + n^{-\lambda}\int_{-1}^{0} x^k \int_{-1}^{0}(-u)^{\lambda+s}\rho^{(s)}(u)\,du\,dx \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{5}$$

where the substitutions $u = nt$ and $v = nx^\mu$ have been made.

It follows immediately that

$$\operatorname*{N\text{-}lim}_{n\to\infty} I_1 = \operatorname*{N\text{-}lim}_{n\to\infty} I_3 = 0 \tag{6}$$

for $k = 0, 1, 2, \ldots$.

Further,

$$\int_1^n v^{(k+1)/\mu-1}(v-u)^{\lambda+s}\,dv = \sum_{i=0}^{\infty}\binom{\lambda+s}{i}(-u)^i\int_1^n v^{(k+1)/\mu+\lambda+s-i-1}\,dv$$
$$= \sum_{i=0}^{\infty}\binom{\lambda+s}{i}(-u)^i\,\frac{n^{(k+1)/\mu+\lambda+s-i}-1}{(k+1)/\mu+\lambda+s-i}$$

and so

$$\operatorname*{N\text{-}lim}_{n\to\infty} I_2 = \frac{1}{\lambda\mu+k+1}\binom{\lambda+s}{s}\int_{-1}^{1}(-u)^s\rho^{(s)}(u)\,du$$
$$= \frac{\Gamma(\lambda+s+1)}{(\lambda\mu+k+1)\Gamma(\lambda+1)} \tag{7}$$

for $k = 0, 1, 2, \ldots.$

It now follows from Eqs. (5)–(7) that

$$\operatorname*{N\text{-}lim}_{n\to\infty}\int_{-1}^{1} x^k\big[\big(x_+^{\mu}\big)_+^{\lambda}\big]_n\,dx = (\lambda\mu+k+1)^{-1}, \tag{8}$$

for $k = 0, 1, 2, \ldots.$

We now consider the case $k = r$, where r is chosen so that $0 < \lambda\mu + r + 1 < 1$, and let ψ be an arbitrary continuous function. Then

$$\frac{\Gamma(\lambda+s+1)}{\Gamma(\lambda+1)}\int_0^{n^{-1/\mu}} x^r\psi(x)\big[\big(x_+^{\mu}\big)_+^{\lambda}\big]_n\,dx$$
$$= \frac{n^{-(\lambda\mu+r+1)/\mu}}{[illegible]}\int_0^1 v^{(r+1)/\mu-1}\int_{-1}^{v}(v-u)^{\lambda+s}\rho^{(s)}(u)\,du\,dv$$

and it follows that

$$\lim_{n\to\infty}\int_0^{n^{-1/\mu}} x^r\psi(x)\big[\big(x_+^{\mu}\big)_+^{\lambda}\big]_n\,dx = 0. \tag{9}$$

When $x \leqslant 0$, we have

$$\frac{\Gamma(\lambda+s+1)}{\Gamma(\lambda+1)}\int_{-1}^{0} x^r\psi(x)\big[\big(x_+^{\mu}\big)_+^{\lambda}\big]_n\,dx = n^{-\lambda}\int_{-1}^{0} x^r\psi(x)\int_{-1}^{0}(-u)^{\lambda+s}\rho^{(s)}(u)\,du\,dx$$

and it follows that

$$\operatorname*{N\text{-}lim}_{n\to\infty}\int_{-1}^{0} x^r\psi(x)\big[\big(x_+^{\mu}\big)_+^{\lambda}\big]_n\,dx = 0. \tag{10}$$

When $x^\mu \geqslant 1/n$, we have

$$\begin{aligned}
\frac{\Gamma(\lambda+s+1)}{\Gamma(\lambda+1)}\big[\big(x_+^\mu\big)_+^\lambda\big]_n &= \int_{-1/n}^{1/n} \big(x^\mu - t\big)^{\lambda+s}\delta_n^{(s)}(t)\,dt \\
&= n^s \int_{-1}^{1} \big(x^\mu - u/n\big)^{\lambda+s}\rho^{(s)}(u)\,du \\
&= n^s x^{(\lambda+s)\mu}\sum_{i=0}^{\infty}\binom{\lambda+s}{i}\int_{-1}^{1}\frac{(-u)^i\rho^{(s)}(u)}{n^i x^{i\mu}}\,du \\
&= \binom{\lambda+s}{s}x^{\lambda\mu} + O\big(n^{-1}\big)
\end{aligned}$$

and so

$$\big[\big(x_+^\mu\big)_+^\lambda\big]_n = x^{\lambda\mu} + O\big(n^{-1}\big). \tag{11}$$

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's theorem we have

$$\varphi(x) = \sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0) + \frac{x^r}{r!}\varphi^{(r)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
\big\langle\big[\big(x_+^\mu\big)_+^\lambda\big]_n, \varphi(x)\big\rangle &= \int_{-1}^{1}\big[\big(x_+^\mu\big)_+^\lambda\big]_n\varphi(x)\,dx \\
&= \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}x^k\big[\big(x_+^\mu\big)_+^\lambda\big]_n\,dx + \int_{n^{-1/\mu}}^{1}\frac{x^r}{r!}\big[\big(x_+^\mu\big)_+^\lambda\big]_n\varphi^{(r)}(\xi x)\,dx \\
&\quad + \int_{0}^{n^{-1/\mu}}\frac{x^r}{r!}\big[\big(x_+^\mu\big)_+^\lambda\big]_n\varphi^{(r)}(\xi x)\,dx + \int_{-1}^{0}\frac{x^r}{r!}\big[\big(x_+^\mu\big)^\lambda\big]_n\varphi^{(r)}(\xi x)\,dx.
\end{aligned}$$

Using Eqs. (8) to (11), it follows that

$$\begin{aligned}
\operatorname*{N\text{-}lim}_{n\to\infty}\big\langle\big[\big(x_+^\mu\big)_+^\lambda\big]_n, \varphi(x)\big\rangle &= \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{(\lambda\mu+k+1)k!} + \int_{0}^{1}\frac{x^{\lambda\mu+r}}{r!}\varphi^{(r)}(\xi x)\,dx \\
&= \int_{0}^{1}x^{\lambda\mu}\Bigg[\varphi(x) - \sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0)\Bigg]dx + \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{(\lambda\mu+k+1)k!} \\
&= \big\langle x_+^{\lambda\mu}, \varphi(x)\big\rangle,
\end{aligned}$$

on using Eq. (1). This proves Eq. (3) on the interval $[-1, 1]$. However, Eq. (3) clearly holds on any interval not containing the origin, and the proof is complete. □

Corollary 6.1. *The distribution $(x_-^\mu)_-^\lambda$ exists and*

$$(x_-^\mu)_-^\lambda = x_-^{\lambda\mu} \tag{12}$$

for $\lambda < 0$, $\mu > 0$ and $\lambda, \lambda\mu \neq -1, -2, \ldots.$

Proof. Equation (12) follows on replacing x by $-x$ in Eq. (3).

Theorem 7. *The distribution $(|x|^\mu)_+^\lambda$ exists and*

$$(|x|^\mu)_+^\lambda = |x|^{\lambda\mu} \tag{13}$$

for $\lambda < 0$, $\mu > 0$ and $\lambda, \lambda\mu \neq -1, -2, \ldots.$

Proof. It follows from Eq. (4) that

$$\frac{\Gamma(\lambda+s+1)}{\Gamma(\lambda+1)}\big[(|x|^\mu)_+^\lambda\big]_n = \begin{cases} \int_{-1/n}^{1/n}(|x|^\mu - t)^{\lambda+s}\delta_n^{(s)}(t)\,dt, & 1/n < |x|^\mu, \\ \int_{-1/n}^{|x|^\mu}(|x|^\mu - t)^{\lambda+s}\delta_n^{(s)}(t)\,dt, & 0 \leqslant |x|^\mu \leqslant 1/n. \end{cases} \tag{14}$$

Since $[(|x|^\mu)_+^\lambda]_n$ is an even function, it follows that

$$\int_{-1}^{1} x^k\big[(|x|^\mu)_+^\lambda\big]_n\,dx = 0 \tag{15}$$

for $k = 1, 3, \ldots.$

If k is even we have

$$\begin{aligned}\frac{\Gamma(\lambda+s+1)}{2\Gamma(\lambda+1)}\int_{-1}^{1} x^k\big[(|x|^\mu)_+^\lambda\big]_n\,dx &= \int_0^{n^{-1/\mu}} x^k \int_{-1/n}^{x^\mu} (x^\mu - t)^{\lambda+s}\delta_n^{(s)}(t)\,dt\,dx \\ &\quad + \int_{n^{-1/\mu}}^{1} x^k \int_{-1/n}^{1/n} (x^\mu - t)^{\lambda+s}\delta_n^{(s)}(t)\,dt\,dx \\ &= I_1 + I_2\end{aligned}$$

and it follows as above that

$$\underset{n\to\infty}{N\text{-}\lim}\int_{-1}^{1} x^k\big[(|x|^\mu)_+^\lambda\big]_n\,dx = 2(\lambda\mu + k + 1)^{-1} \tag{16}$$

for $k = 0, 2, 4, \ldots.$

We now consider the case $k = 2r$, where r is chosen so that $0 < \lambda\mu + r + 2 < 1$, and let ψ be an arbitrary continuous function. Then it follows as above that

$$\lim_{n\to\infty} \int_0^{n^{-1/\mu}} x^{2r}\psi(x)\big[\big(|x|^\mu\big)_+^\lambda\big]_n\,dx = \lim_{n\to\infty} \int_{-n^{-1/\mu}}^{0} x^{2r}\psi(x)\big[\big(|x|^\mu\big)_+^\lambda\big]_n\,dx = 0 \tag{17}$$

and

$$\big[\big(|x|^\mu\big)_+^\lambda\big]_n = |x|^{\lambda\mu} + O\big(n^{-1}\big) \tag{18}$$

if $|x|^\mu \geqslant 1/n$.

Again let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. Then

$$\varphi(x) = \sum_{k=0}^{2r-1} \frac{x^k}{(k)!}\varphi^{(k)}(0) + \frac{x^{2r}}{(2r)!}\varphi^{(2r)}(\xi x),$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
&\big\langle\big[\big(|x|^\mu\big)_+^\lambda\big]_n, \varphi(x)\big\rangle \\
&\quad= \int_{-1}^{1} \big[\big(|x|^\mu\big)_+^\lambda\big]_n\varphi(x)\,dx \\
&\quad= \sum_{k=0}^{r-1} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!}\int_{-1}^{1} x^{2k-1}\big[\big(|x|^\mu\big)_+^\lambda\big]_n\,dx + \sum_{k=0}^{r-1} \frac{\varphi^{(2k)}(0)}{(2k)!}\int_{-1}^{1} x^{2k}\big[\big(|x|^\mu\big)_+^\lambda\big]_n\,dx \\
&\qquad+ \int_{n^{-1/\mu}}^{1} \frac{x^{2r}}{(2r)!}\big[\big(|x|^\mu\big)_+^\lambda\big]_n\varphi^{(2r)}(\xi x)\,dx + \int_{-1}^{-n^{-1/\mu}} \frac{x^{2r}}{(2r)!}\big[\big(|x|^\mu\big)_+^\lambda\big]_n\varphi^{(2r)}(\xi x)\,dx \\
&\qquad+ \int_{0}^{n^{-1/\mu}} \frac{x^{2r}}{(2r)!}\big[\big(|x|^\mu\big)_+^\lambda\big]_n\varphi^{(2r)}(\xi x)\,dx + \int_{-n^{-1/\mu}}^{0} \frac{x^{2r}}{(2r)!}\big[\big(|x|^\mu\big)_+^\lambda\big]_n\varphi^{(2r)}(\xi x)\,dx.
\end{aligned}$$

Using Eqs. (15) to (18), it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\big\langle\big[\big(|x|^\mu\big)_+^\lambda\big]_n, \varphi(x)\big\rangle \\
&\quad= \sum_{k=0}^{r-1} \frac{\varphi^{(2k)}(0)}{(\lambda\mu+2k+1)(2k)!} + \int_{-1}^{1} \frac{|x|^{\lambda\mu+2r}}{(2r)!}\varphi^{(2r)}(\xi x)\,dx \\
&\quad= \int_{-1}^{1} |x|^{\lambda\mu}\left[\varphi(x) - \sum_{k=0}^{r-1}\frac{x^{2k}}{(2k)!}\varphi^{(2k)}(0)\right]dx + \sum_{k=0}^{r-1} \frac{\varphi^{(2k)}(0)}{(\lambda\mu+2k+1)(2k)!} \\
&\quad= \big\langle |x|^{\lambda\mu}, \varphi(x)\big\rangle,
\end{aligned}$$

on using Eq. (2). This proves Eq. (13) on the interval $[-1, 1]$. However, Eq. (13) clearly holds on any interval not containing the origin, and the proof is complete. □

Corollary 7.1. *The distribution* $(|x|^{\mu})_{-}^{\lambda}$ *exists and*

$$\left(|x|^{\mu}\right)_{-}^{\lambda} = |x|^{\lambda\mu} \tag{19}$$

for $\lambda < 0$, $\mu > 0$ *and* $\lambda, \lambda\mu \neq -1, -2, \ldots$.

Proof. Equation (19) follows on replacing x by $-x$ in Eq. (13).

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